Scaling quasi-isometries and wreath products

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1 Definitions and first properties

Recall that we are interested in the following class of maps:

Definition 1.1. Let (X, d_X), (Y, d_Y) be metric spaces. A map $f\colon X \to Y$ is a *quasi-isometry* if there are two constants $C \geqslant 1$, $K \geqslant 0$ such that

(i) $\frac{1}{C} \cdot d_X(x,y) - K \leqslant d_Y(f(x), f(y)) \leqslant C \cdot d_X(x,y) + K$ for all $x, y \in X$;

(ii) $d_Y(y, f(X)) \leqslant K$ for all $y \in Y$.

We usually call f a $(C,K)-$*quasi-isometry*, and we say that X and Y are *quasi-isometric*.

Equivalently, f is a quasi-isometry if it satisfies (i) and there exists $g\colon Y \to X$ such that $d(g \circ f, \mathrm{Id}_X), d(f \circ g, \mathrm{Id}_Y) \leqslant K$ (and then g is called a *quasi-inverse* of f), where the distance between two maps $h_1, h_2\colon X \to Y$ defined over the same metric space is

$$d(h_1, h_2) := \sup_{x \in X} d_Y(h_1(x), h_2(x)).$$

In our context, X and Y will be bounded degree graphs, and in fact Cayley graphs of finitely generated groups. In this setting, it is not hard to prove that a quasi-isometry $f\colon X \to Y$ has uniformly bounded fibers, namely there exists a $P \geqslant 1$ such that $|f^{-1}(\{y\})| \leqslant P$ for any $y \in Y$. Indeed, if $x, x' \in f^{-1}(\{y\})$ and f is a $(C,K)-$quasi-isometry, then

$$d_X(x, x') \leqslant C \cdot (d_Y(f(x), f(x') + K) = CK$$

as $f(x) = f(x') = y$. As X is locally finite, the claim follows.

The aim of "measure-scaling" quasi-isometries is to try to estimate more precisely the size of pre-images of finite subsets.

Definition 1.2. Let X and Y be bounded degree graphs. Let $k > 0$. A quasi-isometry $f\colon X \to Y$ is *quasi-k-to-one* if there exists $C > 0$ such that

$$\big|k|A| - |f^{-1}(A)|\big| \leqslant C \cdot |\partial_Y A|$$

for all finite subsets $A \subset Y$, where $\partial_Y A := \{y \in Y \backslash A : \exists a \in A, y \sim_Y a\}$ is the *boundary* of A in Y.

Informally, being quasi-k-to-one means that the pre-image of a finite subset of the target space is close to have the cardinality of the finite subset multiplied by k. The scaling factor should therefore be thought as a measurement of how far our map is from a bijection. The next theorem, due to Whyte, also serves as a motivation for this viewpoint.

Theorem 1.3 ([Why99] ; [GT22, Proposition 4.1])**.** *Let X and Y be bounded degree graphs. A quasi-isometry $f\colon X \to Y$ is quasi-one-to-one if and only if it lies at bounded distance from a bijection.*

Observe also that between non-amenable spaces, Definition 1.2 is satisfied for any quasi-isometry $f\colon X \to Y$ and any $k > 0$. Indeed, if Y is not amenable, there exists $\varepsilon > 0$ such that $|\partial_Y A| > \varepsilon \cdot |A|$ for any finite subset $A \subset Y$, and thus

$$\begin{aligned}\big|k|A| - |f^{-1}(A)|\big| &\leqslant k|A| + |f^{-1}(A)| \\ &\leqslant (k+P)\cdot |A| \\ &\leqslant \frac{k+P}{\varepsilon}\cdot |\partial_Y A|\end{aligned}$$

where $P \geqslant 1$ is a uniform bound on $|f^{-1}(\{y\})|$ for $y \in Y$. Thus f is quasi-k-to-one. In particular, from this observation and Whyte's theorem, we deduce that any quasi-isometry $X \to Y$ between non-amenable graphs lies at bounded distance from a bijection.

The situation is drastically different for amenable spaces.

Lemma 1.4. *Let X and Y be amenable bounded degree graphs. If $f\colon X \to Y$ is quasi-k-to-one and quasi-k'-to-one, then $k = k'$.*

Proof. As f is quasi-k-to-one, there is $C > 0$ such that

$$\big|k|A| - |f^{-1}(A)|\big| \leqslant C \cdot |\partial_Y A|$$

for all finite subsets $A \subset Y$. In particular, applying this inequality to a Følner sequence $(F_n)_{n\in\mathbb{N}}$ of Y leads to

$$\big|k|F_n| - |f^{-1}(F_n)|\big| \leqslant C \cdot |\partial_Y F_n|$$

for any $n \in \mathbb{N}$, or equivalently

$$\left|k - \frac{|f^{-1}(F_n)|}{|F_n|}\right| \leqslant C \cdot \frac{|\partial_Y F_n|}{|F_n|}$$

for any $n \in \mathbb{N}$. Thus we get that $k = \lim\limits_{n\to\infty} \frac{|f^{-1}(F_n)|}{|F_n|}$, and by the same reasoning, this last limits also equals to k', whence $k = k'$. □

Let us now analyse how scaling maps behave under classical operations.

Proposition 1.5. *Let X, Y and Z be bounded degree graphs. Let $k_1, k_2 > 0$ and let $f, h\colon X \to Y$, $g\colon Y \to Z$ be three quasi-isometries.*

(i) If f is quasi-k_1-to-one and h is at bounded distance from f, then h is quasi-k_1-to-one.

(ii) If f is quasi-k_1-to-one and g is quasi-k_2-to-one, then $g \circ f\colon X \to Z$ is quasi-$k_1 k_2$-to-one.

(iii) If f is quasi-k_1-to-one, then any of its quasi-inverses is quasi-$\frac{1}{k_1}$-to-one.

The proof of Proposition 1.5 requires several intermediate observations.

Lemma 1.6. *Let X be a bounded degree graph.*

(i) For any finite subset $A \subset X$ and any $S \geqslant 0$, one has $|A^{+S}| \leqslant N^S \cdot |A|$, where $N \geqslant 3$ is an integer larger than the maximal degree of a vertex of X.

(ii) For any finite subset $A \subset X$ and any $S \geqslant 0$, there is a constant $R > 0$ (depending only on S and on X) such that $|A^{+S} \backslash A| \leqslant R \cdot |\partial_X A|$.

Proof. *(i)* Fix any $0 \leqslant i \leqslant S-1$. Given any element a of A, the fact that any vertex of X has degree $\leqslant N$ implies that there are at most N^i paths of length i that starts at $a \in A$ and that ends at an element of A^{+i}. Thus

$$\left|A^{+S}\right| \leqslant |A| \cdot \sum_{i=0}^{S-1} N^i \leqslant N^S \cdot |A|$$

as claimed.
(ii) It suffices to note that $A^{+S} \backslash A \subset (\partial_X A)^{+(S-1)}$ and to apply point *(i)*. □

Lemma 1.7. *Let X and Y be two graphs of bounded degree. Let $f\colon X \to Y$ be a quasi-isometry. Then there exists a constant $L > 0$ such that*

$$|\partial_X f^{-1}(A)| \leqslant L \cdot |\partial_Y A|$$

for any finite subset $A \subset Y$.

Proof. Let $C \geqslant 1$, $K \geqslant 0$ be such that f is a $(C, K)-$quasi-isometry. Notice that f sends two adjacent vertices to two vertices at distance $\leqslant C + K$. Let $P \geqslant 1$ be a uniform bound on pre-images of points under f.
Let $A \subset Y$ be finite. Let $x \in \partial_X f^{-1}(A)$. Thus there is $y \in f^{-1}(A)$ such that $d_X(x, y) = 1$. It follows that $d_Y(f(x), f(y)) \leqslant C + K$ and $f(y) \in A$, so $f(x) \in A^{+(C+K)}$, and thus $x \in f^{-1}(A^{+(C+K)})$. Additionally, $x \notin f^{-1}(A)$, so that we have proved

$$\partial_X f^{-1}(A) \subset f^{-1}(A^{+(C+K)}) \backslash f^{-1}(A).$$

Taking cardinalities, it follows that

$$\begin{aligned}|\partial_X f^{-1}(A)| &\leqslant |f^{-1}(A^{+(C+K)}) \backslash f^{-1}(A)| \\ &= |f^{-1}(A^{+(C+K)} \backslash A)| \\ &\leqslant P \cdot |A^{+(C+K)} \backslash A| \\ &\leqslant P \cdot R \cdot |\partial_Y A|\end{aligned}$$

where $R > 0$ is the constant provided by Lemma 1.6*(ii)*, that depends only on Y and the parameters C and K of f. This finishes the proof. □

Proof of Proposition 1.5. (i) Assume that the distance between f and h is at most Q, and observe that this implies that $f^{-1}(B) \subset h^{-1}(B^{+Q})$ and $h^{-1}(B) \subset f^{-1}(B^{+Q})$ for any finite subset $B \subset Y$.
Now, fix any finite subset $A \subset Y$. We first estimate

$$\left|k_1|A| - |h^{-1}(A)|\right| \leqslant \left|k_1|A| - |f^{-1}(A)|\right| + \left||f^{-1}(A)| - |h^{-1}(A)|\right|. \tag{1.1}$$

We then have

$$\begin{aligned} |f^{-1}(A)| - |h^{-1}(A)| &\leqslant |h^{-1}(A^{+Q})| - |h^{-1}(A)| \\ &\leqslant |h^{-1}(A^{+Q} \setminus A)| \\ &\leqslant P \cdot |A^{+Q} \setminus A| \\ &\leqslant P \cdot R \cdot |\partial_Y A| \end{aligned}$$

where $P \geqslant 1$ is a uniform bound on sizes of pre-images of points under h and $R > 0$ is the constant from Lemma 1.6*(ii)*. On the other hand, we have similarly

$$\begin{aligned} |h^{-1}(A)| - |f^{-1}(A)| &\leqslant |f^{-1}(A^{+Q})| - |f^{-1}(A)| \\ &\leqslant |f^{-1}(A^{+Q} \setminus A)| \\ &\leqslant P' \cdot |A^{+Q} \setminus A| \\ &\leqslant P' \cdot R \cdot |\partial_Y A| \end{aligned}$$

where P' is a uniform bound on sizes of pre-images of points under f. Thus, combining (1.1) and the fact that f is quasi-k_1-to-one, we get the existence of $C > 0$ such that

$$\begin{aligned} \left|k_1|A| - |h^{-1}(A)|\right| &\leqslant \left|k_1|A| - |f^{-1}(A)|\right| + \left||f^{-1}(A)| - |h^{-1}(A)|\right| \\ &\leqslant C \cdot |\partial_Y A| + \max(P, P') \cdot R \cdot |\partial_Y A| \\ &= (C + \max(P, P') \cdot R) \cdot |\partial_Y A| \end{aligned}$$

and this proves that h is quasi-k_1-to-one.

(ii) Fix a finite subset $A \subset Z$. By our assumptions, we know there exist $C > 0$ and $D > 0$ such that

$$\begin{aligned} \left|k_1k_2|A| - |(g \circ f)^{-1}(A)|\right| &\leqslant \left|k_1k_2|A| - k_1|g^{-1}(A)|\right| + \left|k_1|g^{-1}(A)| - |f^{-1}(g^{-1}(A))|\right| \\ &\leqslant k_1 \cdot D \cdot |\partial_Z A| + |\partial_Y g^{-1}(A)| \\ &\leqslant k_1 \cdot D \cdot |\partial_Z A| + L \cdot |\partial_Z A| \\ &\leqslant (k_1 \cdot D + L) \cdot |\partial_Z A| \end{aligned}$$

where $L > 0$ is the constant provided by Lemma 1.7. This proves that $g \circ f$ is quasi-k_1k_2-to-one.

(iii) Let $C \geqslant 1$ and $K \geqslant 0$ be such that f and one of its quasi-inverses $\overline{f} \colon Y \to X$ are (C, K)-quasi-isometries, and such that $f \circ \overline{f}$, $\overline{f} \circ f$ lie at distance $\leqslant K$ from Id_Y, Id_X respectively. Fix a finite subset $A \subset X$.
We first rewrite

$$\begin{aligned} \left|\frac{1}{k_1}|A| - |\overline{f}^{-1}(A)|\right| &= \frac{1}{k_1}\left||A| - k_1|\overline{f}^{-1}(A)|\right| \\ &\leqslant \frac{1}{k_1}\left|k_1|\overline{f}^{-1}(A)| - |f^{-1}(\overline{f}^{-1}(A))|\right| + \frac{1}{k_1}\left||f^{-1}(\overline{f}^{-1}(A))| - |A|\right|. \end{aligned}$$

In this sum, we control the first term using that f is quasi-k_1-to-one: there exists a constant $T > 0$ such that

$$\begin{aligned}\left|k_1|\overline{f}^{-1}(A)| - |f^{-1}(\overline{f}^{-1}(A))|\right| &\leqslant T \cdot |\partial_Y \overline{f}^{-1}(A)| \\ &\leqslant T \cdot L \cdot |\partial_X A|\end{aligned}$$

where $L > 0$ is provided by Lemma 1.7 (and depends only on X and the parameters C and K of $\overline{f}$). For the second term, we write

$$\left|f^{-1}(\overline{f}^{-1}(A))| - |A|\right| \leqslant |A^{+K}| - |A| = |A^{+K} \backslash A| \leqslant R \cdot |\partial_X A|$$

since $\overline{f} \circ f$ is at distance $\leqslant K$ from Id_X and where $R > 0$ comes from Lemma 1.6*(ii)* (and depends only on X and the parameters C and K of f). By symmetry, one also gets that

$$|A| - |f^{-1}(\overline{f}^{-1}(A))| \leqslant |(\overline{f} \circ f)^{-1}(A^{+K} \backslash A)| \leqslant P \cdot |A^{+K} \backslash A| \leqslant P \cdot R \cdot |\partial_X A|$$

where $P \geqslant 1$ is a uniform bound on the sizes of pre-images of points under $\overline{f} \circ f$. Combining all these inequalities, it follows that

$$\begin{aligned}\left|\frac{1}{k_1}|A| - |\overline{f}^{-1}(A)|\right| &\leqslant \frac{1}{k_1}\left|k_1|\overline{f}^{-1}(A)| - |f^{-1}(\overline{f}^{-1}(A))|\right| + \frac{1}{k_1}\left||f^{-1}(\overline{f}^{-1}(A))| - |A|\right| \\ &\leqslant \frac{T \cdot L}{k_1}|\partial_X A| + \frac{P}{k_1} \cdot |\partial_X A| \\ &= \frac{T \cdot L + P}{k_1} \cdot |\partial_X A|\end{aligned}$$

and we conclude that $\overline{f}\colon Y \to X$ is quasi-$\frac{1}{k_1}$-to-one. This finishes the proof. □

As an application of these properties:

Corollary 1.8. *Let G be a finitely generated group, and let H_1 and H_2 be two finite-index subgroups of G. Then there exists a measure-scaling quasi-isometry $H_1 \to H_2$, of scaling factor $\frac{[G:H_2]}{[G:H_1]}$.*

In particular, from this corollary, one deduces the following beautiful fact: in a given finitely generated amenable group, if two finite-index subgroups have the same index, then they must be biLipschitz equivalent.

In conclusion: for X an amenable bounded degree graph, we can define its *scaling quasi-isometry group* as

$$\mathrm{QI}_{\mathrm{sc}}(X) := \{\text{measure-scaling quasi-isometries } X \to X\}/\text{bounded distance}.$$

Then, it follows from Lemma 1.4 and Proposition 1.5 that there is a well-defined group morphism

$$\begin{aligned}\mathrm{Sc}\colon \mathrm{QI}_{\mathrm{sc}}(X) &\longrightarrow \mathbb{R}_{>0} \\ f \text{ quasi-}k\text{-to-one} &\longmapsto k\end{aligned}$$

taking any scaling quasi-isometry of X to its scaling factor. We call this map the *scale morphism.*

Definition 1.9. Let X be an amenable bounded degree graph. Its *scaling group*, denoted $\mathrm{Sc}(X)$, is the image of the scale morphism $\mathrm{Sc}\colon \mathrm{QI}_{\mathrm{sc}}(X) \longrightarrow \mathbb{R}_{>0}$.

The scaling group $\mathrm{Sc}(X)$ is a measure-scaling quasi-isometry invariant of the space X. In particular, given a finitely generated group G, its scaling group $\mathrm{Sc}(G)$ is defined as the scaling group of its Cayley graph with respect to an arbitrary finite generating set, and $\mathrm{Sc}(G)$ does not depend on the choice of such a generating set.

The object being introduced, we should keep in mind two central questions about the behaviour of quasi-isometries of a given finitely generated group:

- Is it true that any self-quasi-isometry $G \to G$ is measure-scaling?
- In general, what is $\mathrm{Sc}(G)$?

Let us now give few examples of scaling groups that are known.

Proposition 1.10 ([GT22, Corollary 6.6])**.** *We have $Sc(G) = \mathbb{R}_{>0}$ if:*

(i) G is a Carnot group or a lattice in a Carnot group;

(ii) $G = SOL(\mathbb{R})$ or a lattice in $SOL(\mathbb{R})$;

(iii) $G = BS(1, n)$ for all $n \geqslant 2$.

Moreover, in case (ii) and (iii), $QI(G) = QI_{sc}(G)$, i.e. any self-quasi-isometry of G is measure-scaling.

The scaling group is also known to be smaller in a number of other situations. However, the proof of these facts is in general much harder, because showing a restriction on the scaling group of a space requires to understand precisely the behaviour of all quasi-isometries of the space. Such informations are known only for very specific classes of groups.

Proposition 1.11 ([Dym10])**.** *Let F be a non-trivial finite group. Then one has*

$$Sc(F \wr \mathbb{Z}) = \langle p_1, \dots, p_k \rangle$$

where $p_1, \dots, p_k$ are the prime numbers appearing in the decomposition of $|F|$.

Algebraic and geometric consequences. Let us now outline some nice consequences that can be deduced from the computation of the scaling group of a finitely generated group.

The first one is algebraic and concerns finite-index subgroups of a given finitely generated group.

Corollary 1.12. *Let G be a finitely generated amenable group. If $Sc(G) = \{1\}$, then G has no proper finite-index subgroups isomorphic to itself.*

Proof. If H is a finite-index subgroup isomorphic to G, composing this isomorphism with the natural inclusion $H \hookrightarrow G$ provides a quasi-$\frac{1}{[G:H]}$-to-one self-quasi-isometry $G \to G$ (using Proposition 1.5), and thus $[G : H] = 1$ since $\mathrm{Sc}(G) = \{1\}$, i.e. $H = G$ cannot be proper. $\square$

More generally, the same proof yields to:

Corollary 1.13. *Let G be a finitely generated amenable group. If $Sc(G) = \{1\}$, and if H_1, H_2 are two biLipschitz equivalent finite-index subgroups of G, then $[G : H_1] = [G : H_2]$.*

Even though these two facts are elementary, they illustrate well the interaction between the algebraic structure of a group with its large-scale geometry. From a complete description of its quasi-isometries, one can deduce non-trivial algebraic facts about subgroups of a given group.

The second consequence is geometric, and deals with the difference between quasi-isometries and biLipschitz equivalences. While finite-index extensions of a finitely generated group yields to groups that are quasi-isometric, one can wonder whenever these extensions are in fact biLipschitz equivalent to the initial group. In view of what has been said above in the non-amenable setting, the question is only relevant for amenable groups.

Corollary 1.14. *Let G a finitely generated amenable group, and let H be a group having G as a finite-index subgroup. Then G and H are biLipschitz equivalent if and only if $[H:G] \in Sc(G)$.*

Proof. Suppose that there is a biLipschitz equivalence $H \to G$. Such a map is quasi-one-to-one, and pre-composing it with the natural inclusion $G \hookrightarrow H$ provides a quasi-$\frac{1}{[H:G]}$-to-one quasi-isometry $G \to G$, by Proposition 1.5. Thus $\frac{1}{[H:G]} \in \mathrm{Sc}(G)$, whence $[H:G] \in \mathrm{Sc}(G)$. Conversely, assume that $[H:G] \in \mathrm{Sc}(G)$, and fix a quasi-$[H:G]$-to-one map $f\colon G \to G$. Post-composing it with the natural inclusion $G \hookrightarrow H$ gives a quasi-one-to-one map $G \to H$, and such a map lies at bounded distance from a bijection by Theorem 1.3. The latter is the desired biLipschitz equivalence. □

2 Scaling groups of lamplighters

Definition 2.1. Let A, B be two groups. Their wreath product $A \wr B$ is the group defined by

$$\left(\bigoplus_B A\right) \rtimes B$$

where B acts on the direct sum via

$$(b \cdot f)(b') := f(b^{-1}b')$$

for any $b, b' \in B$ and any $f \in \bigoplus_B A$.

Hence, elements of $A \wr B$ are pairs (f, b) where f is a finitely supported function on B (i.e. $f(b) = e_A$ for all but finitely many $b \in B$) and $b \in B$. The multiplication law is given by

$$(f, b)(f', b') = (f + b \cdot f', bb')$$

for all $f, f' \in \bigoplus_B A$, $b, b' \in B$, where "+" stands for the composition law in the direct sum.

If A is generated by $S = \{a_1, \ldots, a_m\}$ and B is generated by $T = \{b_1, \ldots, b_m\}$, then

$$U := \{(\delta_{a_i}, e_B), (\mathbf{1}, b_j) : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\}$$

generates the wreath product $A \wr B$. The classical interpretation of such generating sets is as follows. First, think of an element $c \in \bigoplus_B A$ as a colouring of the vertices of $\mathrm{Cay}(B, T)$ with colors coming from A, with only finitely many vertices having a non-trivial color (and these vertices form the *support* of c, denoted $\mathrm{supp}(c)$). Second, think of an element $(c, p) \in A \wr B$ as a pair made of a colouring $c \in \bigoplus_B A$ together with an arrow pointing at some vertex $p \in B$. Then, there are two possible moves to go from (c, p) to a neighbouring vertex in $\mathrm{Cay}(A \wr B, U)$:

- either we only move the arrow to a neighbouring vertex in B, and the colouring stays the same;
- or the arrow stays on the vertex where it stands, but changes the color of this vertex, and replaces it with an adjacent color in $\mathrm{Cay}(A, S)$.

Hence to go from (c, p) to another vertex (d, q) in $A \wr B$, the arrow has to move in B from p to q by visiting all vertices of B where c and d differ. At each of these vertices $t \in B$, the color is changed, and goes from $c(t) \in A$ to $d(t) \in A$.

When the lamp group A is finite, we usually call $A \wr B$ a *lamplighter group*.

In this section, our goal is to prove partially the next result, due to Genevois and Tessera:

Theorem 2.2. *Let F_1, F_2 be non-trivial finite groups. Let G and H be amenable finitely presented one-ended groups. Then there exist integers $a, r, s \geqslant 1$ such that $|F_1| = a^r$, $|F_2| = a^s$, and any quasi-isometry $F_1 \wr G \longrightarrow F_2 \wr H$ is quasi-$\frac{s}{r}$-to-one.*

Corollary 2.3. *Let F be a non-trivial finite group and let G be an amenable finitely presented one-ended group. Then any self-quasi-isometry $F \wr G \longrightarrow F \wr G$ is at bounded distance from a bijection. In particular, $Sc(F \wr G) = \{1\}$.*

To achieve the proof of Theorem 2.2, we will take for granted the key rigidity phenomenon, proved in [GT24].

Theorem 2.4 ([GT24, Theorem 1.18])**.** *Let F_1, F_2 be non-trivial finite groups. Let G and H be amenable finitely presented one-ended groups. Let $\varphi\colon F_1 \wr G \longrightarrow F_2 \wr H$ be a quasi-isometry. Then there exists a bijection $\alpha\colon \bigoplus_G F_1 \longrightarrow \bigoplus_H F_2$ and a quasi-isometry $\beta\colon G \to H$ such that φ is at bounded distance from the map*

$$\begin{aligned} \psi\colon F_1 \wr G &\longrightarrow F_2 \wr H \\ (c, p) &\longmapsto (\alpha(c), \beta(p)) \end{aligned}$$

and any quasi-inverse of φ is at bounded distance from the map

$$\begin{aligned} \overline{\psi}\colon F_2 \wr H &\longrightarrow F_1 \wr G \\ (c, p) &\longmapsto (\alpha^{-1}(c), \overline{\beta}(p)) \end{aligned}$$

where $\overline{\beta}\colon H \to G$ is a quasi-inverse of $\beta\colon G \to H$.

Then, combined with the following observation, it will be enough to prove that the induced QI at the level of the base groups $\beta\colon G \to H$ is scaling:

Lemma 2.5. *Let F_1, F_2 be non-trivial finite groups. Let G and H be finitely generated groups. Let $\alpha\colon \bigoplus_G F_1 \longrightarrow \bigoplus_H F_2$ and $\beta\colon G \to H$ be two maps such that*

$$\begin{aligned} \varphi\colon F_1 \wr G &\longrightarrow F_2 \wr H \\ (c, p) &\longmapsto (\alpha(c), \beta(p)) \end{aligned}$$

is a quasi-isometry. If β is quasi-k-to-one for some $k > 0$, then φ is quasi-k-to-one.

Proof. Denote π_G (resp. π_H) the projection of $F_1 \wr G$ (resp. $F_2 \wr H$) onto the second factor G (resp. H). Fix a finite subset $A \subset F_1 \wr G$. Let $\mathcal{C} \subset \bigoplus_H F_2$ be the collection of all elements of $\bigoplus_H F_2$ appearing as first coordinate of an element of A, and for any $c \in \mathcal{C}$, let $A_c \subset A$ be the subset of all elements of A having c as first coordinate. We then have

$$A = \bigsqcup_{c \in \mathcal{C}} A_c$$

and then $|A| = \sum_{c \in \mathcal{C}} |A_c|$, $|\varphi^{-1}(A)| = \sum_{c \in \mathcal{C}} |\varphi^{-1}(A_c)|$. Now, if we set $B_c := \pi_H(A_c)$ for any $c \in \mathcal{C}$, we have $|A_c| = |B_c|$ and $|\varphi^{-1}(A_c)| = |\beta^{-1}(B_c)|$ since $\pi_H \circ \varphi = \beta \circ \pi_G$. We then deduce that

$$\begin{aligned} \big| k|A| - |\varphi^{-1}(A)| \big| &\leqslant \sum_{c \in \mathcal{C}} \big| k|A_c| - |\varphi^{-1}(A_c)| \big| \\ &= \sum_{c \in \mathcal{C}} \big| k|B_c| - |\beta^{-1}(B_c)| \big| \\ &\leqslant C \cdot \sum_{c \in \mathcal{C}} |\partial_H B_c| \end{aligned}$$

where $C > 0$ is the constant coming from the $k-$scalingness of β. It now remains to notice that $\partial_{F_2 \wr H} A$ contains $\bigsqcup_{c \in \mathcal{C}} \{c\} \times \partial_H B_c$ to conclude that

$$\big| k|A| - |\varphi^{-1}(A)| \big| \leqslant C \cdot |\partial_{F_2 \wr H} A|$$

as was to be proved. Thus φ is quasi-k-to-one. □

In our strategy, we are thus reduced to prove that β is scaling. The proof of this fact requires several preliminary observations. The first one is the following.

Proposition 2.6. *With the above notations, fix a quasi-isometry*

$$\begin{aligned} \varphi \colon F_1 \wr G &\longrightarrow F_2 \wr H \\ (c, p) &\longmapsto (\alpha(c), \beta(p)). \end{aligned}$$

Then there exists a constant $Q \geqslant 0$ such that, for any two colourings $c_1, c_2 \in \bigoplus_G F_1$, the Hausdorff distance between $\beta(supp(c_1^{-1}c_2))$ and $supp(\alpha(c_1)^{-1}\alpha(c_2))$ is at most Q.

Proof. Denote $C \geqslant 1$, $K \geqslant 0$ the parameters of φ and of a quasi-inverse $\overline{\varphi}$, that takes the form

$$\begin{aligned} \overline{\varphi} \colon F_2 \wr H &\longrightarrow F_1 \wr G \\ (c, p) &\longmapsto (\alpha^{-1}(c), \overline{\beta}(p)). \end{aligned}$$

Fix $c_1, c_2 \in \bigoplus_G F_1$, and consider a sequence of colourings

$$a_1 = c_1, a_2, \ldots, a_n = c_2$$

such that, for any $1 \leqslant i \leqslant n-1$, a_i and a_{i+1} differ at exactly one point p_i. This way, $\mathrm{supp}(c_1^{-1}c_2) = \{p_1, \ldots, p_{n-1}\}$. Then we have

$$\begin{aligned} d((\alpha(a_i), \beta(p_i)), (\alpha(a_{i+1}), \beta(p_i))) &= d(\varphi(a_i, p_i), \varphi(a_{i+1}, p_i)) \\ &\leqslant C \cdot d_{F_1 \wr G}((a_i, p_i), (a_{i+1}, p_i)) + K \\ &= C + K \end{aligned}$$

which implies that $\alpha(a_i)$ and $\alpha(a_{i+1})$ may only differ on $B(\beta(p_i), C+K)$, for any $1 \leqslant i \leqslant n-1$. It follows that $\alpha(c_1) = \alpha(a_1)$ and $\alpha(c_2) = \alpha(a_n)$ can only differ on

$$\bigcup_{i=1}^{n-1} B(\beta(p_i), C+K) = \bigcup_{i=1}^{n-1} \{\beta(p_i)\}^{+(C+K)} = \{\beta(p_1), \ldots, \beta(p_{n-1})\}^{+(C+K)} = \beta(\mathrm{supp}(c_1^{-1}c_2))^{+(C+K)}$$

and it follows that $\mathrm{supp}(\alpha(c_1)^{-1}\alpha(c_2)) \subset \beta(\mathrm{supp}(c_1^{-1}c_2))^{+(C+K)}$.

Now, the same argument shows that $\mathrm{supp}(c_1^{-1}c_2)$ lies in the $(C+K)-$neighborhood of

$$\overline{\beta}(\mathrm{supp}(\alpha(c_1)^{-1}\alpha(c_2)))$$

so, applying β, we get that $\beta(\mathrm{supp}(c_1^{-1}c_2))$ lies in the $(C(C+K)+K)-$neighborhood of

$$\beta\left(\overline{\beta}(\mathrm{supp}(\alpha(c_1)^{-1}\alpha(c_2)))\right),$$

and the latter is at Hausdorff distance $\leqslant K$ from $\mathrm{supp}(\alpha(c_1)^{-1}\alpha(c_2))$. It follows that $\beta(\mathrm{supp}(c_1^{-1}c_2))$ lies in the $(C(C+K)+2K)-$neighborhood of $\mathrm{supp}(\alpha(c_1)^{-1}\alpha(c_2))$, so we conclude that

$$d_{\mathrm{Haus}}\left(\beta(\mathrm{supp}(c_1^{-1}c_2)), \mathrm{supp}(\alpha(c_1)^{-1}\alpha(c_2))\right) \leqslant C(C+K)+2K$$

and we are done setting $Q := C(C+K)+2K$. □

The key observation is then the following:

Lemma 2.7. *Let $n, m \geqslant 2$ be two integers. Let G, H be infinite groups and let $\varphi\colon \mathbb{Z}/n\mathbb{Z} \wr G \longrightarrow \mathbb{Z}/m\mathbb{Z} \wr H$ be an aptolic quasi-isometry, i.e.*

$$\begin{aligned} \varphi\colon \mathbb{Z}/n\mathbb{Z} \wr G &\longrightarrow \mathbb{Z}/m\mathbb{Z} \wr H \\ (c,p) &\longmapsto (\alpha(c), \beta(p)) \end{aligned}$$

for some bijection α and quasi-isometry $\beta\colon G \to H$. For every quasi-inverse $\overline{\beta}$ of β, there exists $Q \geqslant 0$ such that:

> *For all subsets $A_1 \subset G$ and $Q' \geqslant Q$, $\alpha^{-1}(\mathcal{L}(\beta(A_1)^{+Q'}))$ is a union of cosets of $\mathcal{L}(A_1)$;*
>
> *conversely, for all $A_2 \subset H$ and $Q' \geqslant Q$, $\alpha(\mathcal{L}(\overline{\beta}(A_2)^{+Q'}))$ is a union of cosets of $\mathcal{L}(A_2)$.*

As a consequence, n and m have the same prime divisors.

The proof of Lemma 2.7 requires the next observation:

Lemma 2.8. *Let $n, m \geqslant 2$. Let G, H be infinite groups. Assume that we are given a constant $Q \geqslant 0$ and two maps $\alpha\colon \bigoplus_G \mathbb{Z}/n\mathbb{Z} \longrightarrow \bigoplus_H \mathbb{Z}/m\mathbb{Z}$ such that, for all colourings $c_1, c_2 \in \bigoplus_G \mathbb{Z}/n\mathbb{Z}$ satisfying $\mathrm{supp}(c_1^{-1}c_2) \subset \{p\}$ for some $p \in G$, then $\mathrm{supp}(\alpha(c_1)^{-1}\alpha(c_2)) \subset B_H(\beta(p), Q)$. Then, for any subset $A \subset G$ and any colouring $c \in \bigoplus_G \mathbb{Z}/n\mathbb{Z}$, we have*

$$\alpha(c\mathcal{L}(A)) \subset \alpha(c)\mathcal{L}(\beta(A)^{+Q}).$$

Proof. We argue by induction over $|A|$. The case $|A| = 1$ is true by assumption. Assume that the conclusion holds for a given cardinality and for every colouring. Fix $a \in A$ and $c' \in c\mathcal{L}(A)$. Then there is $c'' \in c\mathcal{L}(A \backslash \{a\})$ such that $c' \in c''\mathcal{L}(\{a\})$, and by the inductive assumption we know that

$$\alpha(c'') \in \alpha(c)\mathcal{L}(\beta(A \backslash \{a\})^{+Q})$$

and that

$$\alpha(c') \in \alpha(c'')\mathcal{L}(\beta(a)^{+Q}).$$

Thus it follows that

$$\alpha(c') \in \alpha(c'')\mathcal{L}(\beta(a)^{+Q}) \subset \alpha(c)\mathcal{L}(\beta(A\backslash\{a\})^{+Q})\mathcal{L}(\beta(a)^{+Q}) = \alpha(c)\mathcal{L}\left(\beta(A\backslash\{a\})^{+Q} \cup \beta(a)^{+Q}\right)$$

and the latter coincides with $\alpha(c)\mathcal{L}\left(\beta(A)^{+Q}\right)$. This concludes the proof. □

Proof of Lemma 2.7. In such a situation, we know from Proposition 2.6 that there is $Q \geqslant 0$ such that

$$d_{\mathrm{Haus}}\left(\beta(\mathrm{supp}(c_1^{-1}c_2)), \mathrm{supp}(\alpha(c_1)^{-1}\alpha(c_2))\right) \leqslant Q.$$

In particular, the assumption of Lemma 2.8 is satisfied. Fix now any finite subset $A_1 \subset G$ and a number $Q' \geqslant Q$. Let $c' \in \alpha^{-1}\left(\mathcal{L}(\beta(A_1)^{+Q'})\right)$ and let $d \in \mathcal{L}(A_1)$. Then one has

$$\alpha(c'd) \in \alpha(c')\mathcal{L}(\beta(A)^{+Q}) \subset \alpha(c')\mathcal{L}(\beta(A)^{+Q'}) = \mathcal{L}(\beta(A)^{+Q'})$$

using Lemma 2.8, so this proves that

$$c'd \in \alpha^{-1}\left(\mathcal{L}(\beta(A)^{+Q'})\right).$$

Thus $\alpha^{-1}(\mathcal{L}(\beta(A)^{+Q'}))$ is stable by multiplication by elements of $\mathcal{L}(A)$, so it must be a union of cosets of $\mathcal{L}(A)$. Moreover this union must be finite since all involved sets are finite.

For the last claim, note that since there is $k \geqslant 1$ and $d_1, \ldots, d_k$ colourings such that

$$\alpha^{-1}(\mathcal{L}(\beta(A)^{+Q'})) = \bigsqcup_{i=1}^{k} d_i\mathcal{L}(A)$$

it follows that

$$m^{|\beta(A)^{+Q}|} = |\mathcal{L}(\beta(A)^{+Q})| = |\alpha^{-1}(\mathcal{L}(\beta(A)^{+Q}))| = \left|\bigsqcup_{i=1}^{k} d_i\mathcal{L}(A)\right| = k \cdot |\mathcal{L}(A)| = k \cdot n^{|A|}$$

and this equality implies that n and m must have the same prime divisors. □

We have now all tools we need to prove Theorem 2.2.

Proof of Theorem 2.2. Fix a quasi-isometry $\varphi\colon F_1 \wr G \longrightarrow F_2 \wr H$, and write it as

$$\begin{aligned}\varphi\colon F_1 \wr G &\longrightarrow F_2 \wr H\\ (c,p) &\longmapsto (\alpha(c), \beta(p))\end{aligned}$$

using Theorem 2.4. Let $C, K \geqslant 0$ be constants such that β and a quasi-inverse $\overline{\beta}$ are $(C,K)-$quasi-isometries and such that $\overline{\beta} \circ \beta, \beta \circ \overline{\beta}$ are within distance K from the identities.

Claim 2.9. *Fix a prime p and let p_1 (resp. p_2) be the $p-$valuation of n (resp. m). There exists a constant $M \geqslant 1$ such that*

$$\left||A| - \frac{p_2}{p_1}|\beta(A)^{+K}|\right| \leqslant M \cdot |\partial_G A|$$

for any finite subset $A \subset G$.

Let $A \subset G$ and set $B := \overline{\beta}(\beta(A)^{+K})^{+K}$. Observe first that

$$A \subset B \subset A^{+(C+3)K}.$$

Indeed, any $a \in A$ is within distance K from $\overline{\beta}(\beta(a))$, with $\beta(a) \in \beta(A)^{+K}$, so any $a \in A$ is in B as well. Conversely, if we fix $b \in B$, then there exists $y \in \beta(A)^{+K}$ such that $d_G(b, \overline{\beta}(y)) \leqslant K$, so there is $a \in A$ such that $d_H(y, \beta(a)) \leqslant K$. Thus one gets

$$\begin{aligned} d_G(b,a) &\leqslant d_G\left(b,\overline{\beta}(y)\right) + d_G\left(\overline{\beta}(y),\overline{\beta}(\beta(a))\right) + d_G\left(\overline{\beta}(\beta(a)),a\right) \\ &\leqslant K + C\cdot d_H(y,\beta(a)) + K + K \\ &\leqslant (C+3)K \end{aligned}$$

whence $b \in A^{+(C+3)K}$. Taking cardinalities and using Lemma 1.6*(ii)*, it follows that

$$|A| \leqslant |B| \leqslant |A^{+(C+3)K}| = |A| + |A^{+(C+3)K} \backslash A| \leqslant |A| + N^{(C+3)K}\cdot|\partial_G A|$$

where $N \geqslant 3$ is an integer larger than the maximal degree of a vertex in the Cayley graph of G. On the other hand, $\alpha^{-1}(\mathcal{L}(\beta(A)^{+K}))$ is a union of cosets of $\mathcal{L}(A)$ (Lemma 2.7), so the cardinality of $\alpha^{-1}(\mathcal{L}(\beta(A)^{+K}))$ must be a multiple of the cardinality of $\mathcal{L}(A)$, i.e. there is $k \geqslant 1$ such that

$$m^{|\beta(A)^{+K}|} = |\mathcal{L}(\beta(A)^{+K})| = |\alpha^{-1}(\mathcal{L}(\beta(A)^{+K}))| = k\cdot|\mathcal{L}(A)| = k\cdot n^{|A|}$$

so, in these two integers, powers of p must be the same, hence there is $E \geqslant 1$ such that

$$p^{p_2|\beta(A)^{+K}|} = E\cdot p^{p_1|A|}.$$

Likewise, since $\alpha(B)$ is a union of cosets of $\mathcal{L}(\beta(A)^{+K})$, there is $F \geqslant 1$ such that

$$p^{p_1|B|} = F\cdot p^{p_2|\beta(A)^{+K}|}$$

from which it follows that

$$|B| = \frac{1}{p_1}\log(F) + \frac{p_2}{p_1}\cdot|\beta(A)^{+K}|.$$

Additionally, we also have

$$\log(F) \leqslant \log(EF) = p_1(|B| - |A|) \leqslant p_1\cdot N^{(C+3)K}|\partial_G A|$$

and hence

$$\frac{p_2}{p_1}|\beta(A)^{+K}| \leqslant |B| \leqslant \frac{p_2}{p_1}|\beta(A)^{+K}| + N^{(C+3)K}\cdot|\partial_G A|.$$

One last triangle inequality finally provides

$$\left||A| - \frac{p_2}{p_1}|\beta(A)^{+K}|\right| \leqslant \left||A| - |B|\right| + \left||B| - \frac{p_2}{p_1}|\beta(A)^{+K}|\right| \leqslant 2N^{(C+3)K}\cdot|\partial_G A|$$

and Claim 2.9 is proved.

Now we show that n and m are powers of a common number. Fix a prime p and let p_1 (resp. p_2) be the $p-$valuation of n (resp. of m). Let $(A_k)_{k\in\mathbb{N}}$ be a Følner sequence of G. Applying Claim 2.9 to A_k, we get

$$\left||A_k| - \frac{p_2}{p_1}|\beta(A_k)^{+K}|\right| \leqslant M\cdot|\partial_G A_k|$$

or equivalently

$$\left|\frac{p_1}{p_2} - \frac{|\beta(A_k)^{+K}|}{|A_k|}\right| \leqslant \frac{p_1}{p_2} \cdot M \cdot \frac{|\partial_G A_k|}{|A_k|}$$

for any $k \in \mathbb{N}$. As $\frac{|\partial_G A_k|}{|A_k|} \to 0$ when $k \to \infty$, it follows that the sequence $\left(\frac{|\beta(A_k)^{+K}|}{|A_k|}\right)_{k\in\mathbb{N}}$ converges to $\frac{p_1}{p_2}$. Likewise, for another prime q, $\left(\frac{|\beta(A_k)^{+K}|}{|A_k|}\right)_{k\in\mathbb{N}}$ converges to $\frac{q_1}{q_2}$, where q_1 (resp. q_2) is the $q-$valuation of n (resp. m). We conclude that there exists a rational $\frac{r}{s}$ such that, for any prime p, the quotient $\frac{p_1}{p_2}$ of the valuations of p in n and m equals $\frac{r}{s}$. This implies that there exists $k \geqslant 1$ such that $n = k^r$, $m = k^s$.

Lastly, we show that β is quasi-$\frac{s}{r}$-to-one. Fix a finite subset $A \subset H$. Then, using the triangle inequality, we write

$$\begin{aligned}\left|\frac{s}{r}|A| - |\beta^{-1}(A)|\right| &\leqslant \left|\frac{s}{r}|A| - \frac{s}{r}|A^{+K}|\right| + \left|\frac{s}{r}|A^{+K}| - \frac{s}{r}|\beta(\beta^{-1}(A))^{+K}|\right| \\ &\quad + \left|\frac{s}{r}|\beta(\beta^{-1}(A))^{+K}| - |\beta^{-1}(A)|\right| \\ &\leqslant \frac{s}{r}|A^{+K}\backslash A| + \frac{s}{r}\left||A^{+K}| - |\beta(\beta^{-1}(A))^{+K}|\right| \\ &\quad + \left|\frac{s}{r}|\beta(\beta^{-1}(A))^{+K}| - |\beta^{-1}(A)|\right| \\ &\leqslant R \cdot |\partial_H A| + \frac{s}{r}\left||A^{+K}| - |\beta(\beta^{-1}(A))^{+K}|\right| + M \cdot |\partial_G \beta^{-1}(A)|\end{aligned}$$

where $R > 0$ is the constant of Lemma 1.6*(ii)* and $M > 0$ is the constant of Claim 2.9. Now, by Lemma 1.7, we know there is $L > 0$ such that $|\partial_G \beta^{-1}(A)| \leqslant L \cdot |\partial_H A|$, so our previous estimate becomes

$$\left|\frac{s}{r}|A| - |\beta^{-1}(A)|\right| \leqslant \frac{s}{r} \cdot R \cdot |\partial_H A| + \frac{s}{r}\left|A^{+K}\backslash\beta(\beta^{-1}(A))^{+K}\right| + M \cdot L \cdot |\partial_H A|.$$

Lastly, it is an easy observation to notice that $A^{+K}\backslash\beta(\beta^{-1}(A))^{+K} \subset (\partial_H A)^{+(2K-1)}$, hence

$$\left|A^{+K}\backslash\beta(\beta^{-1}(A))^{+K}\right| \leqslant \left|(\partial_H A)^{+(2K-1)}\right| \leqslant P^{2K-1}|\partial_H A|$$

by Lemma 1.6*(i)*, where $P \geqslant 3$ is an integer larger than the maximal degree of a vertex in the Cayley graph of H. Hence we conclude that

$$\left|\frac{s}{r}|A| - |\beta^{-1}(A)|\right| \leqslant \left(\frac{s}{r} \cdot R + \frac{s}{r} \cdot P^{2K-1} + M \cdot L\right)|\partial_H A|$$

and thus β is quasi-$\frac{s}{r}$-to-one. The fact that φ is quasi-$\frac{s}{r}$-to-one follows from Lemma 2.5. □

We have a similar result for wreath products with polynomial growth lamp groups:

Theorem 2.10 ([Dum25, Theorem B])**.** *Let N and M be polynomial growth groups, with growth degrees n and m respectively. Let G and H be finitely presented amenable groups from $\mathcal{M}_{exp}$. Then any quasi-isometry $N \wr G \longrightarrow M \wr H$ is quasi-$\frac{m}{n}$-to-one.*

Corollary 2.11. *Let N be a polynomial growth group, and G be an amenable finitely presented group from $\mathcal{M}_{exp}$. Then any quasi-isometry $N \wr G \longrightarrow N \wr G$ is at bounded distance from a bijection. In particular, $Sc(N \wr G) = \{1\}$.*

3 An application to the QI rigidity of iterated wreath products

From our previous work, we can deduce an arithmetic obstruction to the existence of a quasi-isometry between certain iterated wreath products:

Proposition 3.1 ([Dum25, Proposition G]). *Let $n, m \geqslant 2$. Let N_1, N_2 be polynomial growth groups with growth degrees n_1, n_2 respectively. Let G and H be amenable finitely presented groups from $\mathcal{M}_{exp}$. If there exists a quasi-isometry*

$$\mathbb{Z}/n\mathbb{Z} \wr (N_1 \wr G) \longrightarrow \mathbb{Z}/m\mathbb{Z} \wr (N_2 \wr H)$$

then there exist integers $a, r, s \geqslant 1$ such that $n = a^r$, $m = a^s$ and $\frac{s}{r} = \frac{n_2}{n_1}$.

Proof. The aptolicity phenomenon of Theorem 2.2 also holds for lamplighters over some infinitely presented groups, see [GT24a]. Then our quasi-isometry φ can be taken to be aptolic:

$$\begin{aligned} \varphi \colon \mathbb{Z}/n\mathbb{Z} \wr (N_1 \wr G) &\longrightarrow \mathbb{Z}/m\mathbb{Z} \wr (N_2 \wr H) \\ (c, p) &\longmapsto (\alpha(c), \beta(p)) \end{aligned}$$

for some bijection α and quasi-isometry $\beta \colon N_1 \wr G \longrightarrow N_2 \wr H$. From [GT24a, Theorem 8.6], there are integers $a, r, s \geqslant 1$ such that $n = a^r$, $m = a^s$ and β is quasi-$\frac{s}{r}$-to-one. Additionally, from Theorem 2.10, β is quasi-$\frac{n_2}{n_1}$-to-one, so it follows from Lemma 2.5 that φ is both quasi-$\frac{s}{r}$-to-one and quasi-$\frac{n_2}{n_1}$-to-one. From Lemma 1.4, we conclude that $\frac{s}{r} = \frac{n_2}{n_1}$, as claimed. □

For instance, this corollary allows to rule out the existence of a quasi-isometry

$$\mathbb{Z}/2\mathbb{Z} \wr (\mathbb{Z}^2 \wr \mathrm{BS}(1, n)) \longrightarrow \mathbb{Z}/4\mathbb{Z} \wr (\mathbb{Z}^3 \wr \mathrm{BS}(1, n))$$

for $n \geqslant 2$.

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